PROBLEMS AND SOLUTIONS.

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PROBLEMS FOR SOLUTION.

ALGEBRA.

450. Proposed by J. E. ROWE, Pennsylvania State College.

If the four roots of the quartic equation $A \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$, are so related that $B \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = 0$, show by elementary algebra that two roots of A are real and two imaginary. Show also by means of elementary algebra that A cannot have two equal roots without having three, if the condition $B = 0$ is satisfied.

451. Proposed by H. S. UHLER, Yale University.

Prove that

$$\frac{\sin x}{x} = \cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots.$$

GEOMETRY.

481. Proposed by PAUL CAPRON, U. S. Naval Academy.

Show that the locus of the intersection of a pair of perpendicular normals to a parabola $y^2 = 4px$ is the parabola $y^2 = p(x - 3p)$.

482. Proposed by ROBERT G. THOMAS, The Citadel, Charleston, S. C.

In laying out a kite-shaped mile race-track, composed of a circular arc and two intersecting tangents at the ends of the arc, determine the angle at the center of the arc (*a*) when the length of the arc equals the sum of the two tangents, and (*b*) when the arc is equal to the length of each tangent.

CALCULUS.

402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2) if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x\,\partial y} = 0,$$

where (x', y') are the current coördinates of points on the tangents.

NOTE.—In an early issue, we will publish all the unsolved problems in Number Theory proposed from January, 1913, to December, 1915. EDITORS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

439. Proposed by A. M. KENYON, Purdue University.

If k, n are natural numbers, $n > 2k$, show that

$$\frac{2^k}{\underline{|k}}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\frac{1}{\underline{|2i+1}\,\underline{|n-k-2i}} = \frac{2^n}{\underline{|n+1}}\sum_{i=0}^{k}\binom{n-i}{n-k},$$

where $I(n/2)$ denotes the integral part of $n/2$ and $\binom{n}{k}$ is the coefficient of x^k in $(1+x)^n$.

SOLUTION BY FRANK IRWIN, University of California.

On the right side

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\binom{n+1}{n-k+1}.$$

This formula expresses in symbols the well-known property of the Pascal triangle, that any term is equal to the sum of all terms above it in the preceding column. (See, for instance, LUCAS, *Théorie des Nombres*, page 6.)

Or it may be proved as follows:

$$\sum_{i=0}^{k}\binom{n-i}{n-k}=\sum_{i=0}^{k}\left[\binom{n-i+1}{n-k+1}-\binom{n-i}{n-k+1}\right]=\sum_{i=0}^{k}\binom{n-i+1}{n-k+1}-\sum_{i=1}^{k+1}\binom{n-i+1}{n-k+1}.$$

Here all the terms but one cancel in pairs, leaving $\binom{n+1}{n-k+1}$.

On the other hand, the left side of our given equation may be written

$$\frac{2^k}{\underline{|n-k+1}\;\underline{|k}}\sum_{i=0}^{I\left(\frac{n+1}{2}\right)}\binom{n-k+1}{2i+1}.$$

Here the summation sign takes in the 2d, 4th, $\cdots$, *all* the even-placed coefficients,

$$\binom{n-k+1}{1},\quad\binom{n-k+1}{3},\ \cdots$$

in the development of $(1+x)^{n-k+1}$; the sum of which is well known to be 2^{n-k}.

Our formula, then, reduces to the obviously true form:

$$\frac{1}{\underline{|n-k+1}}\frac{2^k}{\underline{|k}}2^{n-k}=\frac{2^n}{\underline{|n+1}}\binom{n+1}{n-k+1},$$

Also solved by the Proposer.

440. Proposed by W. D. CAIRNS, Oberlin College.

n being a positive integer, find the sum of the series

$$2n^2+4(n-1)^2+2(n-2)^2+4(n-3)^2+2(n-4)^2+\cdots,$$

where the succeeding coefficients are alternately 4 and 2; or, more generally, the series

$$an^2+b(n-1)^2+a(n-2)^2+b(n-3)^2+a(n-4)^2+\cdots.$$

L'Intermédiaire, July, 1913.

SOLUTION BY ELIJAH SWIFT, University of Vermont.

The problem as originally published in the September, 1915, issue contained two misprints. The question is indefinite. These series are not convergent, and do not break off after a finite number of terms. However, it is easy to find the sum of $2k$ terms.

Expanding, we can write the second series as

$$\begin{aligned}S&=a\{n^2+n^2-4n+4+n^2-8n+16+n^2-12n+36+\cdots+n^2-2(2k-2)n+(2k-2)^2\}\\&\quad+b\{n^2-2n+1+n^2-6n+9+n^2-10n+25+\cdots+n^2-2(2k-1)n+(2k-1)^2\}\\&=a\{kn^2-4n(1+2+3+\cdots+k-1)+(2^2+4^2+6^2+\cdots+[2k-2]^2)\}\\&\quad+b\{kn^2-2n(1+3+5+\cdots+2k-1)+(1^2+3^2+5^2+\cdots+[2k-1]^2)\}.\end{aligned}$$